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ABSTRACT

Given the current and widespread practical interest in mathematical understanding, particularly with respect to higher order thinking skills, curriculum reform advocates in many countries cite the need for teaching mathematics with understanding. However, the characterization of understanding in ways that highlight its growth, as well as the identification of pedagogical actions that sponsor understanding, represent continuing problem areas. The focus of this report is a dynamic, multidimensional, multidirectional model for the theory of the growth of understanding within a specific individual on a specific topic. Background to this theory is provided in the companion document, entitled "A Dynamic Theory of Mathematical Understanding: Some Features and Implications." Included is a discussion of various classroom situations surrounding the topic of quadratic equations that illustrate the eight embedded levels of understanding within the model, consisting of: (1) primitive knowing; (2) image making; (3) image having; (4) property noticing; (5) formalizing; (6) observing; (7) structuring; and (8) inventizing. These situational differences illustrate the various teacher-student interactions that guide the student through the eight embedded levels of understanding through the following specific processes: image doing, image reviewing, image seeing, image saying, property predicting, and property recording. (JJK)

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THE CHARACTERISTICS OF THE GROWTH OF MATHEMATICAL UNDERSTANDING

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THE CHARACTERISTICS OF THE GROWTH OF MATHEMATICAL UNDERSTANDING
AN ORAL PRESENTATION AT AERA 1991, CHICAGO

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Welcome to our oral presentation, which, as you see from our pre-paper[1], will focus on the "Dynamic Theory of the Growth of Mathematical Understanding". True to that title, to what we propose to do, and to the program, we will be looking at the dynamics in the growth of mathematical understanding, particularly with respect to the inner kinds of understanding acts: image making, image having and property noticing.

Notice that we have 'dynamic' modifying 'theory' in the title of the paper. A title which truly captures the essence of our thoughts would be "The Dynamic Theory of the Dynamic Growth of Dynamic Mathematical Understanding" ! We work within the philosophy of "theory as process' and what we will do today will reflect our very recent and new theoretical thinking and research.

Our previous work focussed on building the aspects and concepts of our theory and supporting them with examples culled from observations made during our own personal, unrelated work with people doing mathematics. We tried to identify people in the acts of understanding as suggested by our theory. Work on this problem is reflected in the paper you have and in the bibliography at the end of it.We are not going to go over the background or focus on that previous work today. But we have the diagram and the terms we use in our work on the poster[2] to remind you of the ideas.

The work reported on today reflects a new focus. Following on from building and exemplifying the theory, we now wish to use it to analyse the on-going mathematical activity of students and to try to observe and characterise the results of understanding. It is important that you see, that we see understanding as a process and not as an acquisition or location.. We will focus here on the six verbs, doing, reviewing, seeing, saying, predicting and recording which form the acting/expressing complementarities within the image making, image having and property noticing rings. We will not be looking at the theory in general which is covered in the pre-paper, nor will we be looking at the outer four rings.

We will try in the first part of the session to give you more detailed characterisations of these complementary acts of 'acting' and 'expressing' within the three inner rings of our model. [Remember these are acts and not acquisitions, and the rings illustrate modes of understanding rather than outwardly monotonic phases.] We will then try to illustrate these acts through analysis of understanding actions pertinent to a growing understanding of quadratic functions. We will also look at the potential impacts of interventions by a teacher during the process of growth.

Time, in this brief presentation, will not permit us to discuss other uses we are making of the theory to study understanding behaviours, nor to tie our work to other constructivist theoretical work.

Before we get to the task outlined above, we need to make a substantive digression. This digression we hope will give you some insight into what we had to do to come up with the kind of definitions and distinctions in our theory which would not only allow us to look for, identify, and describe behaviours which confirmed (for us) the existence of elements of the theory, but also would allow you to see the dynamics of on-going behaviour through the lens of this theory.

We were really struggling to define behaviour that seemed to us near the "boundaries" between our rings and as we looked at the mathematical behaviour of children, real or supposed, we found that we knew "too much" about the maths that children were doing. Hence we had difficulty thinking about the minimum criteria for observing any of our theoretical behaviours. Fortunately, we ourselves were in the throes of trying to understand how we could use E-mail to aid us in our joint writing, which usually evolves from faxes and discs flying across the Atlantic. Neither of us is very proficient with E-mail, so we tried to think how our model might be used to

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[1]The paper refered to is 'A Dynamic Theory of Mathematical Understanding: some features and implications'.

[2]A reduced copy is attached to this paper

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look at our understanding activities with respect to E-mail and to think about the growth and change of this understanding. We are neither of us what you might call computer literate! Bear with us those of you who are.

From our own experience, we could see that image making (our inner ring) for understanding E-mail entailed doing the activity of fulfilling a list of given actions, such as

"shift, return, return. Type vax, Type se0793n, type nmail,....."

Further, our own theory (and theories of others such as von Glasersfeld, Steffe, Cobb and Maturana, Varela, and Tomm) allowed us to see that image having involved being able to do E-mail not just through our own personal machines, but on other systems as well. Understanding was not just tied by action in a particular context. But what wouldn't come clear was the pathway from that initial crude doing to this more portable scheme.

The technique which solved the problem was not a new one, but its power was well illustrated for us in this case. We asked ourselves, "How would understanding behaviours manifest themselves, not when the person was 'right' or 'satisfied' with an act, but when they were 'wrong' or 'dissatisfied' with the outcome of their activity?" When, for example, one of us could not send e-mail on a different machine.

Let us look at this in detail. We might be able to engage in the process aspect of image making for E-mail. That is, we could follow directions and send a message, but to say we exhibited image making understanding, the aspect which we are calling the 'expressing' of image making was necessary'to complete the image making understanding act. It did no good to simply say that this 'expressing' aspect entailed seeing the E-mail sending process as a whole. While that is true, it is hardly the minimal characterisation we sought. We needed to ask "What if going through the listed actions to send E-mail 'failed' to work for me?"

Now if we had not been 'expressing' our image making we would likely quit or start all over again. If we could 'express' our image making understanding we would likely review what had been done and say,"Oh, I see - I pushed the wrong key in the entry five lines back." Such doing and reviewing would now characterise image making understanding for E-mail, and although such actions might seem simple, such understanding would allow a person to send and receive E-mail in a particular location quite successfully.

To observe the transition to image having, we tried the same 'negative trick.' From observing our own experience, we thought that one thing having an image of E-mail would mean, would be that we would expect to be able to send an E-mail in a new environment - from someone else's machine.

But what of the transition to this state? Imagine our coming to a different computer and trying to access and read an E-mail message. If our effort, even with review, failed, what might we do? If we thought, "I guess you can't do E-mail here" then we could not be said to be image having. Although we could use E-mail in our own environment, we were not showing image having behaviour. Our procedure was not portable.If, however, we said, "Well, E-mail must work differently here," and asked for help, perhaps in terms of a new list of instructions, we would be engaging in the *process* of image having. We would have seen that there was a pattern to E-mail , even if we could not enunciate it or didn't know the precise details. Of course, such "seeing" is fragile. To express image having, that is, to show complete image having behaviour, we would have to notice and enunciate features of E-mail. In the face of an unsuccessful E-mail experience, we might say, "I must have made an error in logging on", or "I clearly need a different command to access the directory". "I should be able to do these things in e-mail, I just don't know how on this computer."

Before leaving this bit of our history in developing a characterisation of these levels and moving on to mathematical understanding per se, we should reiterate a couple of points. Firstly, our previous definitions of the types and interrelationships between understanding behaviours - as seen in the embedded levels of our model - were sufficient to allow us to see examples of the various behaviours in children's and adult's activities in mathematics at all levels - right through from image making to structuring. Indeed they were sufficient to allow us to analyse the nature of the mathematical understanding being exhibited in the argument between Krantz and Mandelbrot in the July 1990 *Mathematical Intelligencer*. However to generate the minimal conditions definitions. to allow us to observe dynamic transitions. the behavioural responses in the

face of lack of success were important. It was easy to find positive instances of people working at a particular level, but we needed to study negative cases to sharpen up our boundary definitions.

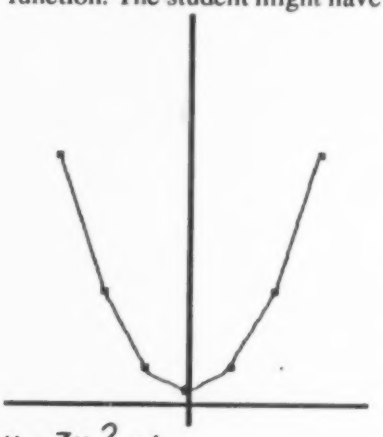
If you have followed our previous work, then looking at our current labels you will see that we have made some changes. In order to more clearly talk about students' understanding actions, we have decided to use one name for each of our rings and names different from those for the acting and expressing complements within each ring. Our analyses of growth of understanding, in order to minimally characterise these actions, has influenced our new choice of the verbs.

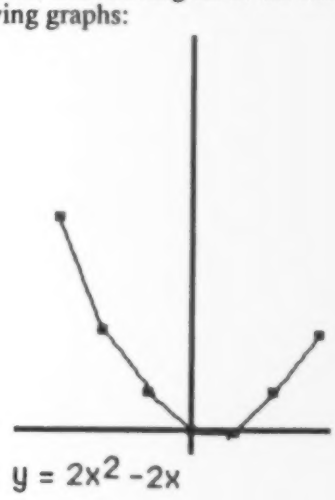
We now intend to trace with you the dynamics of the growth of understanding of a mathematical topic (by a person) through the three pairs of acting and expressing complementarities using our latest analyses and terminology. We have elected to return to a topic which we analysed previously in our article in *For The Learning Of Mathematics* in 1989 - namely quadratic functions. We wish to make it quite clear that we do NOT claim that the teaching sequence discussed below is ideal, merely that it illustrates our analysis. The teacher presenting the initial task below might assume that the student's primitive knowing would include: evaluating polynomial expressions (at least of the second degree), making tables of values, and graphing points from these tables. The observed 'acting' aspect of image making, which we are calling image doing would entail responding to the following task:

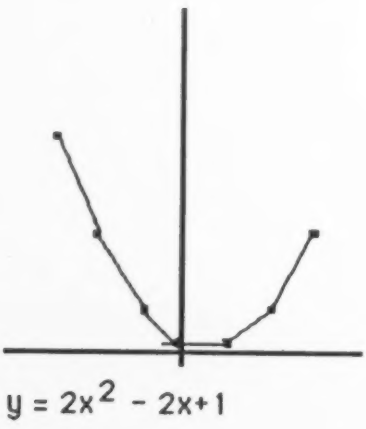
Consider the function $y= 3x^2 + 1$
Make a table of values for x and y , x taking values from -3 to +3
Now draw the graph of the function.

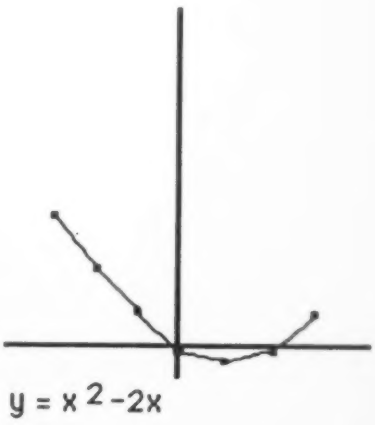
Repeat this for the following functions :- $y=x^2$, $y=2x^2-2x$, $y=2x^2-2x+1$, $y=x^2-2x$

.In showing the image doing 'acting' behaviour for quadratic functions, a student might be observed making such a table of values, plotting and joining the points, and moving on to the next function. The student might have successfully completed the following graphs:







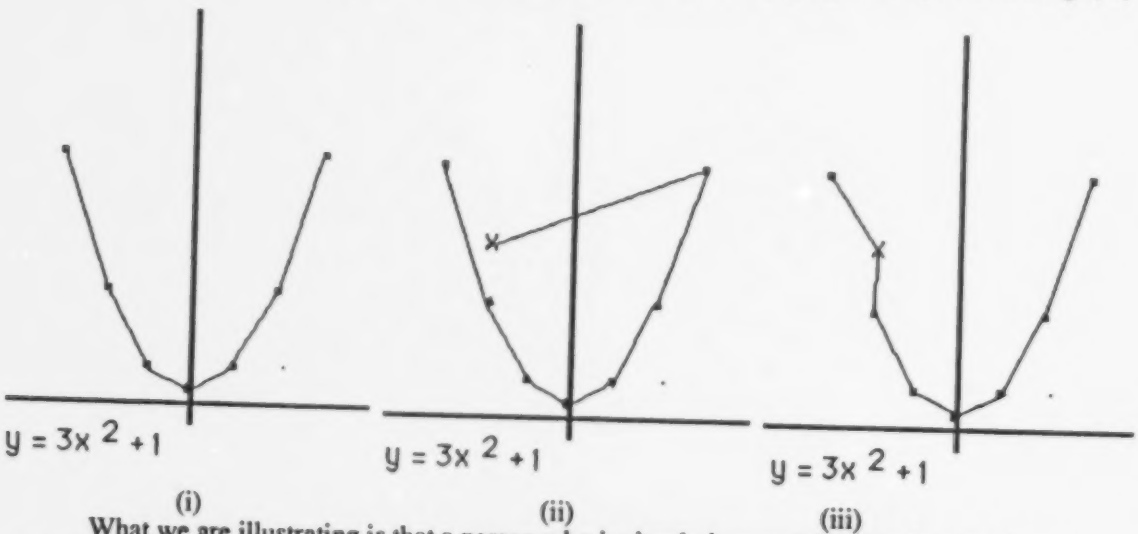


(Please note that we are not recommending this as the way to teach quadratic functions, merely illustrating our theory)

In such behaviour one could not see if the student had done the 'expressing' behaviour of image reviewing, even though the student might have considered the graph as a whole before moving on. We wish to say here (and have collected evidence to support the assertion) that to understand a mathematical topic even by showing image making behaviour, a student must have done image reviewing as well as image doing. Image doing is not enough for understanding.

The best way for us to show the distinctions between 'image doing', 'image reviewing', and the image having behaviours of 'image seeing' and image saying' is to run through with you scenarios of pupil-teacher interactions.

Situation 1: The teacher adds to the student's graph a point, say (-2,20) which is within the range plotted (-3 to +3) but not on the drawn line and asks the pupil whether it belongs to her graph.
If the student simply rejects the point as "not part of my graph", (i), or joins it to the last point plotted, (ii), then the student is still image doing.
If the student rubs out the appropriate joining line,here this would be the line joining (-3,28) and (-2,13),and connects in the new point, then the student would be engaging in Image Reviewing. (iii)



(i) (ii) (iii)

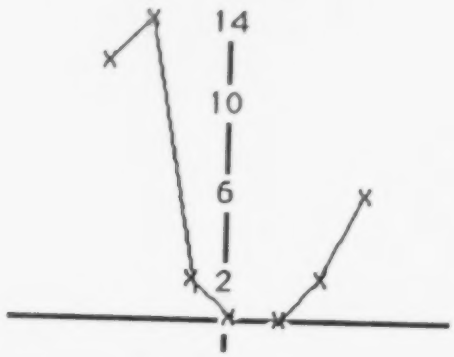
What we are illustrating is that a person who is simply image doing sees her previous action as completed and rejects returning to it. The image reviewing behaviour allows for the constructive alteration of previous behaviour without yet seeing a pattern. The teacher's intervention was intended to be provocative - deliberately trying to move the student to image reviewing, but, of course, it is the student's response to the situation that determines the nature of the questions for that person. In the case of the pupil who rejects the new point the question is serving to validate that the pupil is still at the image doing stage.

Situation 2: The teacher produces a table of values for the student to plot, which contains a point that should not be on the graph .
If the student merely plots the points and joins them up in order then, as above, she is image doing.
If she says "That seems odd" but does nothing.then she is Image Reviewing.
If, however, she says "I would have thought that the point should be here. I'll check your computations because it shouldn't look like that.", then her understanding is at an Image Seeing level.

Here the students' responses to erroneous plotting might show or validate their level. Hence teachers might deliberately give such a table of values to validate their hunch as to the kind of behaviour the student is showing (here at the boundary between image reviewing and image seeing).

x	-3	-2	-1	0	1	2	3
x^2	9	12	1	0	1	4	9
-x	3	2	1	0	-1	-2	-3
y	12	14	2	0	0	2	6

$y = x^2 - x$



Situation 3: The teacher adds an outlying point which would obviously - to those who HAVE an image - not be part of the quadratic function and asks the student its relevance to her graph.

The student might "correctly"join the new point to the rest of the curve if she were Image Reviewing.

If she were Image Seeing then we would expect a response on the lines of "I would have thought this wouldn't have a point like that." If she had, however, moved to the level of Image Saying, them she would be making statements like "That is wrong. These are all pointy or flat-bottomed U shapes."Notice the distinction between image seeing and image saying. The former - seeing - has "collected together" previous instances and has a pattern. The image saying behaviour articulates its features.

This person is in a position to talk about her actions and carry them beyond the graphing situation.You might also notice that, even for these inner understandings, there are ways by which students can be expected to judge and defend their behaviour. They have "organised" even these very informal schemes.

The particular example we are using was deliberately done to suggest that image having does not imply having the "right " or complete picture. Given the image expressed - that they are all U-shaped - a teacher might ask questions with an invocative intention - to tempt the student to fold back to further image making activity - further individual graphing. For example "Have you tried to graph $y=1-x^2$?". Such new image doing and reviewing would now be informed (or at least affected by) the image so far seen and articulated.

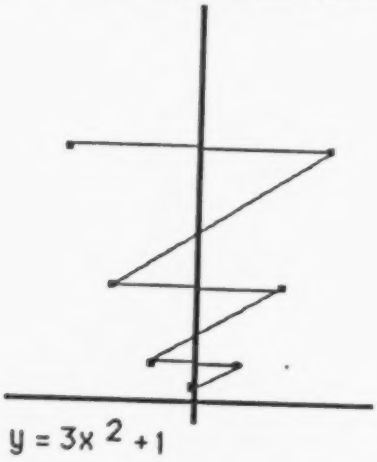
To this point we have tried to make distinctions between image doing and reviewing, between image seeing and image saying, and between the activities of image making and image having generally. We are saying that a person showing both image doing and reviewing is showing a kind of understanding in their actions and we are also saying that a person who is image having is engaging in a qualitatively different kind of understanding activity, in seeing and saying that quadratics are not simply the successful results of graphing activity, but are things with identifiable features. Now, if a person thinks "A quadratic which has an odd-numbered value as the coefficient of x^2 is pointy and an even-numbered one is flat-bottomed," based on the graphs illustrated here, they are engaging in the property noticing activity of property prediction. They are connecting and explaining features of their image - a new kind of understanding activity. To complete property noticing, the person must engage in property recording. We have observed in our research several instances of where students have engaged in property predicting without recording (at least making note that a property existed and seemed to "work"). Such lack of 'expressing' activity seemed to inhibit the students from moving beyond their previous image.

Notice again that we have selected a property - pointy bottoms and flat bottoms governed by the coefficient of x^2 - which is not a 'usual' property of quadratics and, in fact, may later be proven wrong or incomplete. We are looking at the nature of understanding as an activity and not as a particular content. The teacher, in a situation such as that just described, could be expected to provide the student with the opportunity to defend her 'property' by testing it against new instances which are deliberately chosen to invoke folding back, further image making and further property noticing behaviour to allow the student to adjust and extend her image. We've said all that before, but our new distinction now allows us to comment in finer detail on a couple of teaching incidents.

Firstly, consider the situation where a student has an image for quadratic equations as U-shaped and the teacher says,"This is a property of quadratics: The line of symmetry for a quadratic function is the line

$y = {}^{-\text{coeff of } x}/_{2 \text{ coeff of } x^2}$ This will be on the test tomorrow, so I thought I'd better tell you." If the student just wrote this down and memorised it for use, he would have recorded, but not predicted a property. Whether or not he managed to remember and use it in the test, until that student engaged in further activity, it would be apart from his current understanding. He would have a disjoint or broken understanding.

Secondly, let's look back at our initial activity.

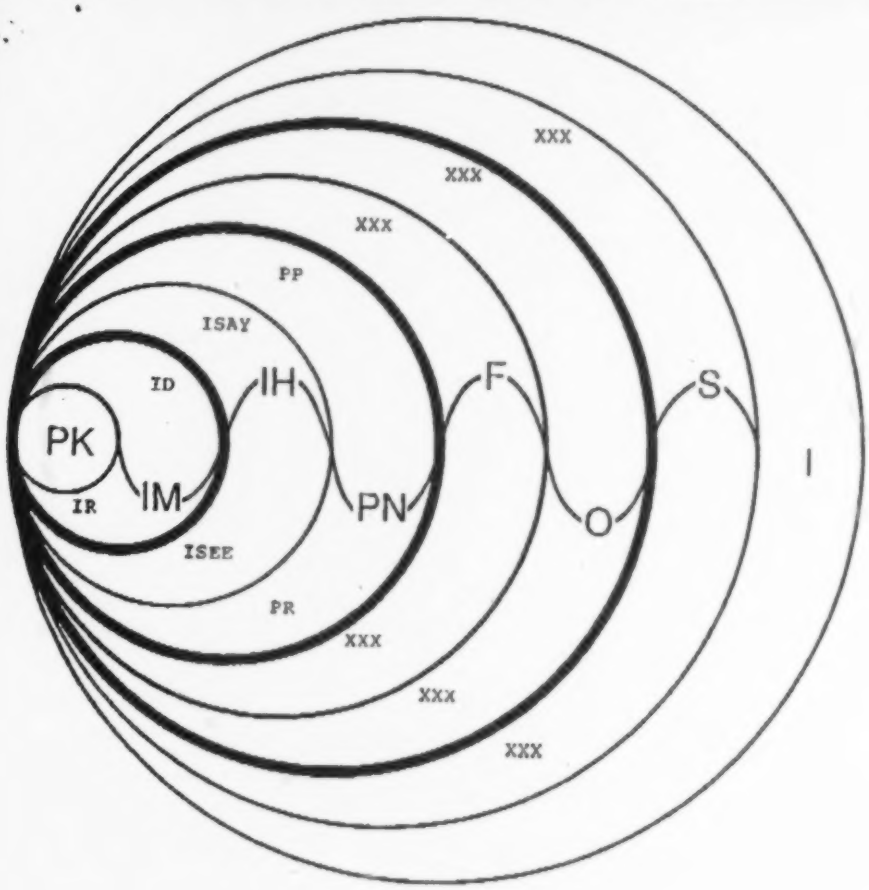


Suppose students were told to make a table of values, but not necessarily told to join the graph in the order of the x values. They may engage in a task which is congruent with their primitive knowing, but is not even image doing with respect to quadratic functions. They are engaged in inappropriate doing

As we are wont to do at this point in our discussion, we cite Maturana :

"Either there is much more to say or nothing."

We have much more to say theoretically and from our analysis of classroom activity in mathematics. But we think it is time for you to act and talk. so let's look at a brief transcript together, through the eyes of our theory.



PK: Primative Knowing
IM: Image Making
IH: Image Having
PN: Property Noticing
F: Formalizing
O: Observing
S: Structuring
I: Inventising

ID : Image Doing
IR : Image Reviewing
ISEE : Image Seeing
ISAY : Image Saying
PP : Property Predicting
PR : Property Recording

Key Words

"Don't Need" Boundary
Complementary
Embedded
Folding Back
Provocative
Invocative
Validating